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On The Use of "Small External Fields" in The Problem of Symmetry Breakdown in Statistical Mechanics*

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We study the infinite system equilibrium states in the statistical mechanics of classical lattice gases. We show that breakdown of the translation invariance occurs if and only if the derivative $dP(\Phi + \lambda\Psi)/d\lambda$ is discontinuous at $\lambda = 0$ for some Ψ. In this formula, P is the pressure, Φ the translation invariant interaction of the system, and Ψ a "small external field" from a suitable class of nontranslation invariant interactions. In an appendix we show that an Ising ferromagnet in a nonvanishing magnetic field has only one equilibrium state.

INTRODUCTION

The equilibrium states for an infinite system in statistical mechanics may be defined as the thermodynamic limits of finite system equilibrium states.[1] The latter are given by the grand canonical ensemble with various boundary conditions. If the interaction Φ is translation invariant, there is at least one (translation) invariant equilibrium state, but there may be several (corresponding to several thermodynamic phases). Even if there is only one invariant equilibrium state (phase), there may be many equilibrium states, forming a convex set. Every equilibrium state ρ is the barycenter of a unique measure μ carried by the extremal equilibrium states (μ gives the unique decomposition of ρ into pure equilibrium states). In particular, let there be a unique invariant state ρ (pure phase); if the support of the corresponding measure μ is not reduced to $\{\rho\}$ we say that *the translational invariance of the theory is broken.* This *symmetry breakdown* implies that there exist nontranslationally invariant equilibrium states. On the other hand, it is not known if the absence of symmetry breakdown implies that there is only one equilibrium state. It would be very interesting to have a proof, or a counterexample.

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1 For details see Dobrushin [2] and Lanford and Ruelle [8].

The idea of much work on ferromagnets, antiferromagnets, superfluids, etc. ... is that the breakdown of the translational symmetry can be detected by use of a small external field $\lambda\Psi$. Let $P(\Phi)$ be the thermodynamic pressure corresponding to the interaction Φ. It is expected that $dP(\Phi + \lambda\Psi)/d\lambda$ is continuous in λ at $\lambda = 0$, for all Ψ of a suitable class, if and only if there is no symmetry breakdown, i.e., if and only if there is a unique invariant equilibrium state ρ which cannot be decomposed into noninvariant equilibrium states.

In the present note we restrict our attention to the simple case of classical lattice gases. The external fields Ψ are nontranslationally invariant interactions such that $P(\Phi + \lambda\Psi)$ is well defined. To satisfy this last requirement we introduce a class $\mathscr{B}^A$ of *averageable* interactions Ψ which generalize the *random* interactions of Griffiths and Lebowitz [6]. Our main result is then Theorem 2 which expresses that the differentiability of $P(\Phi + \lambda\Psi)$ at $\lambda = 0$ for all averageable Ψ is equivalent to the absence of symmetry breakdown: the unique invariant equilibrium state ρ is an extremal equilibrium state.

1. Notation

We indicate here briefly the notation and results which will be used.

If E is a set, $|E|$ is the cardinal of E, $\mathscr{P}(E)$ is the set of subsets of E, $\mathscr{P}_f(E)$ is the set of finite subsets of E.

An *interaction* for a classical lattice gas (on a lattice $\mathbf{Z}^\nu$) is a function Φ: $\mathscr{P}_f(\mathbf{Z}^\nu) \to \mathbf{R}$ such that $\Phi(\phi) = 0$. We restrict ourselves here to bounded interactions, for which

$$\|\Phi\| = \sum_{X \ni 0} \sup_{x \in \mathbf{Z}^\nu} |\Phi(X + x)| < +\infty.$$

These interactions form a Banach space $\mathscr{B}^B$ containing as subspace $\mathscr{B}$ the translation invariant interactions.

The topology of pointwise convergence of the characteristic functions makes $\mathscr{P}(\mathbf{Z}^\nu)$ into a compact set. We call $\mathfrak{A}$ the abelian C^*-algebra of complex continuous functions on this set. If $\Lambda \subset \mathbf{Z}^\nu$, we let $\mathfrak{A}_\Lambda$ be the subalgebra of functions A such that

$$A(X) = A(X \cap \Lambda).$$

A *state* σ is a probability measure on $\mathscr{P}(\mathbf{Z}^\nu)$, or, equivalently, a positive linear form on $\mathfrak{A}$ such that $\sigma(1) = 1$.

Let $\Phi \in \mathscr{B}^B$. If $X \in \mathscr{P}_f(\mathbf{Z}^\nu)$, we write

$$U_\Phi(X) = \sum_{S \subset X} \Phi(S).$$

If, furthermore, $Y \subset \mathbf{Z}^\nu$ and $X \cap Y = \phi$, we let

$$W_\Phi(X, Y) = \sum_{S \subset X \cup Y}^{*} \Phi(S),$$

where $\sum^*$ extends to those S such that $S \cap X \neq \phi$ and $S \cap Y \neq \phi$. A state σ is an equilibrium state with respect to Φ if, for every $\Lambda \in \mathscr{P}_f(\mathbf{Z}^\nu)$, there exists a probability measure $\tilde{\sigma}_\Lambda$ on $\mathscr{P}(\mathbf{Z}^\nu/\Lambda)$ such that

$$\sigma(A) = \int \tilde{\sigma}_\Lambda(dY) \frac{\sum_{X \subset \Lambda} A(X) \exp[-U_\Phi(X) - W_\Phi(X, Y)]}{\sum_{X \subset \Lambda} \exp[-U_\Phi(X) - W_\Phi(X, Y)]} \tag{1}$$

for all $A \in \mathfrak{A}_\Lambda$. The equations (1) are called *equilibrium equations* (see [2]). For every $\Phi \in \mathscr{B}^B$ there exists at least one equilibrium state.

When $x \in \mathbf{Z}^\nu$, $X \subset \mathbf{Z}^\nu$, we introduce the translated set $X + x$. If $A \in \mathfrak{A}$, we define a translated function $\tau_x A$ by

$$\tau_x A(X) = A(X - x).$$

Finally, if $a = (a^1, \dots, a^\nu)$ and $a^1, \dots, a^\nu$ are strictly positive integers, we let

$$\Lambda(a) = \{x \in \mathbf{Z}^\nu : 0 \leqslant x^i < a^i\}$$

and write $a \to \infty$ for $a^1, \dots, a^\nu \to \infty$.

2. Averageable Interactions

Lemma. *The ball* $B_r = \{\Phi \in \mathscr{B}^B : \| \Phi \| \leqslant r\}$ *is compact with respect to the topology* $\mathscr{T}$ *of pointwise convergence* (of functions $\mathscr{P}_f(\mathbf{Z}^\nu) \to \mathbf{R}$).

In the space of all functions $\mathscr{P}_f(\mathbf{Z}^\nu) \to \mathbf{R}$, B_r is the intersection of the closed sets

$$B_{X_1 \cdots X_n} = \left\{\Phi : \sum_{i=1}^{n} | \Phi(X_i)| \leqslant r\right\},$$

where $(X_1, \dots, X_n)$ is any finite family of translates of sets $X_1^0, \dots, X_n^0$, all distinct and containing 0. Furthermore, B_r is contained in the compact set

$$\{\Phi : | \Phi(X)| \leqslant r\}$$

product of one copy of the interval $[-r, r]$ for each $X \in \mathscr{P}_f(\mathbf{Z}^\nu)$.

Definition. Let $\Phi \in \mathscr{B}^B$ and $\| \Phi \| \leqslant r$. We say that Φ is *averageable* if, for every a with strictly positive integer components,

$$\lim_{b \to \infty} | \Lambda(b)|^{-1} \sum_{n \in \Lambda(b)} \delta(\tau_{-na}\Phi) \tag{2}$$

exists in the vague topology of measures on $B_r(\mathscr{T})$.[2] We have used the notation

$$\tau_x\Phi(X) = \Phi(X - x),$$
$$\delta(\Phi) = \text{unit mass at } \Phi,$$
$$na = (n^1a^1,..., n^\nu a^\nu).$$

THEOREM 1. (a) *The set* $\mathscr{B}^A$ *of averageable interactions is closed in* $\mathscr{B}^B$ *and invariant under translations of* $\mathbf{Z}^\nu$. If $\Phi \in \mathscr{B}$, $\Psi \in \mathscr{B}^A$, *and* $\lambda \in \mathbf{R}$, *then* $\Phi + \lambda\Psi \in \mathscr{B}^A$.

(b) *If* $\Phi \in \mathscr{B}^A$, *the limit*

$$P(\Phi) = \lim_{a\to\infty} |\Lambda(a)|^{-1} \log \sum_{X\subset\Lambda(a)} \exp[-U_\Phi(X)]$$

exists. $P(\cdot)$ *is continuous, convex on* $\mathscr{B}^A$, *and translation invariant. More precisely, if* $\Phi, \Psi \in \mathscr{B}^A$, *then*

$$| P(\Phi) - P(\Psi)| \leqslant \| \Phi - \Psi \|.$$

If, furthermore, $0 < \alpha < 1$, *and* $\alpha\Phi + (1 - \alpha)\Psi \in \mathscr{B}^A$, *then*

$$P(\alpha\Phi + (1 - \alpha)\Psi) \leqslant \alpha P(\Phi) + (1 - \alpha) P(\Psi).$$

Finally, if $x \in \mathbf{Z}^\nu$,

$$P(\tau_x\Phi) = P(\Phi).$$

The proof of (a) is immediate. To prove (b) we use the density of finite range interactions. We say that $\Psi \in \mathscr{B}^B$ has uniformly finite range if there exists $\Delta \in \mathscr{P}_f(\mathbf{Z}^\nu)$ such that $X - X \subset \Delta$ whenever $\Psi(X) \neq 0$. Let $\mathscr{B}_0{}^B$ be the space of these interactions. If $\Psi \in \mathscr{B}^B$ and $\Lambda \in \mathscr{P}_f(\mathbf{Z}^\nu)$, we let

$$\Psi_\Lambda(X) = \begin{cases} \Psi(X) & \text{when } X - X \subset \Lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\lim_{\Lambda\to\infty} \Psi_\Lambda = \Psi$, showing that $\mathscr{B}_0{}^B$ is dense in $\mathscr{B}^B$. If Ψ is averageable, one checks readily that Ψ_Λ is averageable, so that $\mathscr{B}_0{}^A = \mathscr{B}^A \cap \mathscr{B}_0{}^B$ is dense in $\mathscr{B}^A$. Let now

$$P_\Lambda(\Phi) = | \Lambda |^{-1} \log \sum_{X\subset\Lambda} \exp[-U_\Phi(X)].$$

[2] More explicitly, (2) means that for every continuous (complex) function f on $B_r(\mathscr{T})$ the following limit exists:

$$\lim_{b\to\infty} | \Lambda(b)|^{-1} \sum_{n\in\Lambda(b)} f(\tau_{-na}\Phi).$$

It suffices to check the existence of this limit for sufficiently large a.

With the same arguments as for translation invariant interactions [9, Section 2.3], we find that if $\Phi \in \mathscr{B}_0{}^A$, given $\epsilon > 0$ there exists a such that

$$\left| P_{\Lambda(ab)}(\Phi) - \Lambda\, |(b)|^{-1} \sum_{n \in \Lambda(b)} P_{\Lambda(a)+na}(\Phi) \right| < \epsilon$$

for all b. On the other hand,

$$\lim_{b\to\infty} |\, \Lambda(b)|^{-1} \sum_{n\in\Lambda(b)} P_{\Lambda(a)+na}(\Phi)$$

$$= \lim_{b\to\infty} |\, \Lambda(b)|^{-1} \sum_{n\in\Lambda(b)} P_{\Lambda(a)}(\tau_{-na}\Phi)$$

exists because Φ is averageable. The proof of (b) proceeds then as in the case of translationally invariant interactions.

Remarks. (a) In Theorem 1(a) one can let Λ tend to infinity in a more general manner, provided the definition of averageable interactions is suitably modified.

(b) If $\Lambda \to \infty$ on an ultrafilter, then, for all $\Phi \in \mathscr{B}^B$, we may write

$$\lim P_\Lambda(\Phi) = P(\Phi).$$

With this definition, P is continuous, convex, and translation invariant, but non-unique.

3. A Characterization of Symmetry Breakdown

Lemma. *Given $\Lambda \in \mathscr{P}_f(\mathbf{Z}^\nu)$, we consider the projection $p_{\setminus\Lambda}$: $\mathscr{P}(\mathbf{Z}^\nu) \to \mathscr{P}(\mathbf{Z}^\nu \setminus \Lambda)$ and for any state φ let $\varphi_{\setminus\Lambda}$ be the image of φ by this projection:*

$$\varphi_{\setminus\Lambda} = p_{\setminus\Lambda}\varphi.$$

Let $\Phi \in \mathscr{B}^A$. If we define

$$P_{\varphi,\Lambda}(\Phi) = |\, \Lambda\,|^{-1} \int \varphi_{\setminus\Lambda}(dY) \log \sum_{X\subset\Lambda} \exp[-U_\Phi(X) - W_\Phi(X, Y)],$$

we have

$$\lim_{a\to\infty} P_{\varphi,\Lambda(a)}(\Phi) = P(\Phi) \tag{3}$$

and, if φ is an equilibrium state for the interaction Φ,

$$\lim_{a\to\infty} \left[|\, \Lambda(a)|^{-1}\, \varphi(U_{\Psi,\Lambda(a)}) + \frac{\partial}{\partial\lambda} P_{\varphi,\Lambda(a)}(\Phi + \lambda\Psi)\Big|_{\lambda=0} \right] = 0, \tag{4}$$

where $\Psi \in \mathscr{B}^A$ and $U_{\Psi,\Lambda}(X) = U_\Psi(X \cap \Lambda)$.

Part (3) follows from standard estimates; (4) follows from the equilibrium equations and the relation

$$\lim_{a\to\infty}\Big[|\Lambda(a)|^{-1}\int \varphi_{\backslash\Lambda(a)}(dY)\frac{\sum_{X\subset\Lambda(a)} U_\Psi(X)\exp[-U_\Phi(X)-W_\Phi(X,Y)]}{\sum_{X\subset\Lambda(a)}\exp[-U_\Phi(X)-W_\Phi(X,Y)]} - |\Lambda(a)|^{-1}$$

$$\times\int \varphi_{\backslash\Lambda(a)}(dY)\frac{\sum_{X\subset\Lambda(a)} [U_\Psi(X)+W_\Psi(X,Y)]\exp[-U_\Phi(X)-W_\Phi(X,Y)]}{\sum_{X\subset\Lambda(a)}\exp[-U_\Phi(X)-W_\Phi(X,Y)]}\Big] = 0.$$

THEOREM 2. *Let $\Phi\in\mathscr{B}$. The following conditions are equivalent.*

(a) *There is only one translation invariant equilibrium state ρ, and it is an extremal equilibrium state.*

(b) *If ρ is an invariant equilibrium state, σ any equilibrium state, and $A = A^*\in\mathfrak{A}$, then*

$$\lim_{a\to\infty}|\Lambda(a)|^{-1}\sum_{x\in\Lambda(a)}[\sigma(\tau_x A)-\rho(A)]^2 = 0 \tag{5}$$

uniformly in σ.

(c) *For each $\Psi\in\mathscr{B}^A$ (or $\Psi\in\mathscr{B}_0{}^A$), the function $\lambda\to P(\Phi+\lambda\Psi)$ is differentiable at zero.*

First, we assume (a), and prove (b). Define $\sigma[\Lambda]$ as an average over translations

$$\sigma[\Lambda](A) = |\Lambda|^{-1}\sum_{x\in\Lambda}\sigma(\tau_x A).$$

We claim that

$$\lim_{a\to\infty}\sigma[\Lambda(a)](A) = \rho(A). \tag{6}$$

In the opposite case there would exist a limit point σ^* of the $\sigma[\Lambda(\mathrm{a})]$ such that $\sigma^*(A)\neq\rho(A)$, and σ^* is an equilibrium state invariant under translations, against the assumptions. Let $\Lambda'(a) = \{x\in\Lambda(a): \sigma(\tau_x A) > \rho(A)\}$ and $\Lambda''(a) = \Lambda(a)\backslash\Lambda'(a)$. If $\sigma[\Lambda'(a)]$, $\sigma[\Lambda''(a)]$, $|\Lambda'(a)|/|\Lambda(a)|$ tend to σ'^*, σ''^*, λ, we have, by (6),

$$\lambda\sigma'^* + (1-\lambda)\,\sigma''^* = \rho.$$

If (5) did not hold uniformly with respect to σ, we could arrange that

$$|\Lambda(a)|^{-1}\sum_{x\in\Lambda(a)}[\sigma(\tau_x A)-\rho(A)]^2\to C>0,$$

and therefore,

$$\lambda[\sigma'^*(A) - \rho(A)] - (1 - \lambda)[\sigma''^*(A) - \rho(A)]$$
$$= \lim |\Lambda(a)|^{-1} \sum_{x \in \Lambda(a)} |\sigma(\tau_x A) - \rho(A)| \geqslant C/2 \| A \| > 0$$

in contradiction with the extremality of ρ.

To show that (b) implies (c), we let (λ_k) be chosen such that $\lambda_k \to 0$, and the function $\lambda \to P(\Phi + \lambda\Psi)$ is differentiable at λ_k. Also let σ_k be an equilibrium state for each interaction $\Phi + \lambda_k\Psi$. Assuming (b), we prove the differentiability of $P(\Phi + \lambda\Psi)$ at $\lambda = 0$ by showing that the derivative at λ_k has a unique limit when $k \to \infty$. This results from the following relations

$$-\frac{\partial}{\partial\lambda} P(\Phi + \lambda\Psi)\Big|_{\lambda=\lambda_k} = -\lim_{a\to\infty} \frac{\partial}{\partial\lambda} P_{\sigma_k,\Lambda(a)}(\Phi + \lambda\Psi)\Big|_{\lambda=\lambda_k}$$
$$= \lim_{a\to\infty} |\Lambda(a)|^{-1} \sigma_k(U_{\Psi,\Lambda(a)}), \tag{7}$$

$$\lim_{k\to\infty} \lim_{a\to\infty} |\Lambda(a)|^{-1} \sigma_k(U_{\Psi,\Lambda(a)}) = \lim_{a\to\infty} |\Lambda(a)|^{-1} \rho(U_{\Psi,\Lambda(a)}). \tag{8}$$

Equation (7) follows from (4). It remains thus to prove (8) and, by density, it suffices to do this for $\Psi \in \mathscr{B}_0{}^A$.

Given $\Psi \in \mathscr{B}_0{}^A$ and $\epsilon > 0$, it follows from (b) that a exists such that

$$|\Lambda(a)|^{-1} |\sigma(U_{\Psi,\Lambda(a)+na}) - \rho(U_{\Psi,\Lambda(a)+na})| < \epsilon \tag{9}$$

for all $n \in \mathbf{Z}^\nu$ and all equilibrium states σ. We may also assume that, for all b,

$$|\Lambda(ab)|^{-1} \Big| U_{\Psi,\Lambda(ab)} - \sum_{n\in\Lambda(b)} U_{\Psi,\Lambda(a)+na} \Big| < \epsilon. \tag{10}$$

For Λ sufficiently large, $\Lambda \supset \Lambda(a)$, the function

$$A \to \Big(\sum_{X\subset\Lambda} e^{-U_\Phi(X)-W_\Phi(X,Y)}\Big)^{-1} \sum_{X\subset\Lambda} A(X)\, e^{-U_\Phi(X)-W_\Phi(X,Y)}$$

on $\mathfrak{A}_{\Lambda(a)}$ is arbitrarily close to the restriction of some equilibrium state to $\mathfrak{A}_{\Lambda(a)}$, uniformly in $Y \subset \mathbf{Z}^\nu\backslash\Lambda$. Inserting this in (9) gives

$$|\Lambda(a)|^{-1} \Big| \int \tilde\sigma_{kn}(dY) \frac{\sum_{X\subset\Lambda+na} U_{\Psi,\Lambda(a)+na}(X)\, e^{-U_\Phi(X)-W_\Phi(X,Y)}}{\sum_{X\subset\Lambda+na} e^{-U_\Phi(X)-W_\Phi(X,Y)}} - \rho(U_{\Psi,\Lambda(a)+na})\Big| < 2\epsilon,$$

where $\tilde\sigma_{kn}$ is any probability measure on $\mathscr{P}(\mathbf{Z}^\nu\backslash(\Lambda + na))$. In this relation, we replace the interaction Φ in the exponentials by $\Phi + \lambda_k\Psi$. For sufficiently large k

this causes little change. Taking then $\tilde{\sigma}_{kn} = p_{\setminus(\Lambda+na)}\,\sigma_k$ and applying the equilibrium equations gives

$$|\Lambda(a)|^{-1}\,|\,\sigma_k(U_{\Psi,\Lambda(a)+na}) - \rho(U_{\Psi,\Lambda(a)+na})| < 3\epsilon.$$

Finally, using (10) we find

$$|\Lambda(ab)|^{-1}\,|\,\sigma_k(U_{\Psi,\Lambda(ab)}) - \rho(U_{\Psi,\Lambda(ab)})| < 5\epsilon,$$

and hence

$$\lim_{c\to\infty} |\Lambda(c)|^{-1}\,|\,\sigma_k(U_{\Psi,\Lambda(c)}) - \rho(U_{\Psi,\Lambda(c)})| \leqslant 5\epsilon,$$

which proves (8), and therefore (b) $\Rightarrow$ (c).

We prove (c) $\Rightarrow$ (a) by assuming that (a) does not hold and showing that, for suitable $\Psi \in \mathscr{B}_0{}^A$, the function $\lambda \to P(\Phi + \lambda\Psi)$ is not differentiable at zero. If there are more than one translation invariant equilibrium states, this can be achieved with $\Psi \in \mathscr{B}$[9]. If there is only one invariant equilibrium state ρ, let μ_ρ be the unique measure with resultant ρ carried by the extremal equilibrium states [8]. If $A \in \mathfrak{A}$ we let $\hat{A}$ be the function on states defined by $\hat{A}(\sigma) = \sigma(A)$. By assumption μ_ρ is not carried by $\{\rho\}$, and we can choose $A = A^*$ such that $\mu_\rho(\hat{A}^2) \neq \rho(A)^2$. We may also suppose that $A \in \mathfrak{A}_{\Lambda(a)}$ for some a, and that $A(\Phi) = 0$. For μ_ρ-almost all σ we have

$$\begin{aligned} &\lim_{b\to\infty} |\Lambda(b)|^{-1} \sum_{n\in\Lambda(b)} \sigma(\tau_{na}A) = \mu_\rho(\hat{A}) = \rho(A) \\ &\lim_{b\to\infty} |\Lambda(b)|^{-1} \sum_{n\in\Lambda(b)} [\sigma(\tau_{na}A)]^2 = \mu_\rho(\hat{A}^2). \end{aligned} \tag{11}$$

Let $\Psi_\sigma(X)$ vanish unless X is contained in a translate $\Lambda(a) + na$ of $\Lambda(a)$, and let

$$|\Lambda(a)|^{-1} \sum_{Y\subset X} \Psi_\sigma(Y) = \sigma(\tau_{na}A)\cdot\tau_{na}A(X) \tag{12}$$

for all $X \subset \Lambda(a) + na$. This defines Ψ_σ unambiguously and we note that, by the pointwise ergodic theorem, Ψ_σ is averageable for μ_ρ-almost all σ. We fix now σ such that $\Psi = \Psi_\sigma \in \mathscr{B}_0{}^A$ and (11) holds. Inserting (12) into (4) we get

$$\begin{aligned} -\lim_{b\to\infty} \frac{\partial}{\partial\lambda} P_{\varphi,\Lambda(ab)}(\Phi + \lambda\Psi_\sigma)\Big|_{\lambda=0} &= \lim_{b\to\infty} |\Lambda(b)|^{-1}\,\varphi\Big(\sum_{n\in\Lambda(b)} \sigma(\tau_{na}A)\,\tau_{na}A\Big) \\ &= \lim_{b\to\infty} |\Lambda(b)|^{-1} \sum_{n\in\Lambda(b)} \sigma(\tau_{na}A)\,\varphi(\tau_{na}A). \end{aligned}$$

In particular for $\varphi = \rho$ the r.h.s. is $\rho(A)^2$, for $\varphi = \sigma$ it is $\mu_\rho(\hat{A}^2)$, and, since these are different, (3) shows that $\lambda \to P(\Phi + \lambda\Psi)$ is not differentiable at zero.

Remark. One can use Theorem 2 to prove that there is no breakdown of translational invariance in an Ising ferromagnet when the magnetic field is different from zero. In this case one can, however, show that there is only one equilibrium state. The proof of this stronger result is presented in the Appendix.

Appendix

THEOREM. *Let Φ be a pair interaction on $\mathbf{Z}^\nu$: $\Phi(X) = 0$ for $|X| > 2$, $\Phi(\{x, y\}) = \beta\varphi(x - y)$, $\Phi(\{x\}) = -\beta\mu$. If $\varphi \leqslant 0$ and*

$$\mu \neq \tfrac{1}{2} \sum_{x \neq 0} \varphi(x), \tag{A1}$$

then there is only one equilibrium state associated with the interaction Φ. (In the equivalent language of spin systems we may say that an Ising ferromagnet has only one equilibrium state when the magnetic field is different from zero.)

First we prove some lemmas.

LEMMA 1. *Let Λ be a finite subset of $\mathbf{Z}^\nu$. We denote by ρ_Λ^+, ρ_Λ^-, and ρ_Λ the correlation functions of a system contained in Λ with the following boundary conditions: all the points of $\mathbf{Z}^\nu\backslash\Lambda$ are occupied (ρ_Λ^+), or all the points of $\mathbf{Z}^\nu\backslash\Lambda$ are empty (ρ_Λ^-), or some are occupied and some are empty (ρ_Λ). Then, if $X \subset \Lambda \subset \Lambda'$,*

$$\rho_\Lambda^-(X) \leqslant \rho_\Lambda(X) \leqslant \rho_\Lambda^+(X) \tag{A2}$$

$$\rho_\Lambda^-(X) \leqslant \rho_{\Lambda'}^-(X), \qquad \rho_{\Lambda'}^+(X) \leqslant \rho_\Lambda^+(X). \tag{A3}$$

If Y is the set of occupied points in $\mathbf{Z}^\nu\backslash\Lambda$, we have

$$\rho_\Lambda(X) = \frac{\sum_{S:X\subset S\subset\Lambda} \exp[-U(S) - W(S, Y)]}{\sum_{S\subset\Lambda} \exp[-U(S) - W(S, Y)]}$$

$$= \lim_{M\to\infty} \frac{\sum_{S:X\subset S\subset\Lambda} \exp[-U(S \cup (Y \cap M))]}{\sum_{S\subset\Lambda} \exp[-U(S \cup (Y \cap M))} .$$

Let now $\epsilon_Y(T) = |T \cap Y| - |T\backslash Y|$, then

$$\rho_\Lambda(X) = \lim_{M\to\infty} \lim_{\lambda\to+\infty} \rho_{\Lambda M\lambda}^Y(X),$$

$$\rho_{\Lambda M\lambda}^Y(X) = \frac{\sum_{S:X\subset S\subset\Lambda} \sum_{T\subset M\backslash\Lambda} \exp[-U(S \cup T) + \lambda\epsilon_Y(T)]}{\sum_{S\subset\Lambda} \sum_{T\subset M\backslash\Lambda} \exp[-U(S \cup T + \lambda\epsilon_Y(T)]} .$$

A theorem by Fortuin, Ginibre and Kasteleyn [3], extending a theorem by Griffiths,[3] asserts that the correlation function for a lattice gas with negative pair interaction depends monotonically on the chemical potential at each lattice point. Therefore,

$$\rho^{\phi}_{\Lambda M\lambda}(X) \leqslant \rho^{Y}_{\Lambda M\lambda}(X) \leqslant \rho^{\mathbf{Z}^\nu}_{\Lambda M\lambda}(X)$$

$$\rho^{\phi}_{\Lambda M\lambda}(X) \leqslant \rho^{\phi}_{\Lambda' M\lambda}(X), \qquad \rho^{\mathbf{Z}^\nu}_{\Lambda' M\lambda}(X) \leqslant \rho^{\mathbf{Z}^\nu}_{\Lambda M\lambda}(X)$$

and the lemma follows by letting $\lambda \to +\infty$, $M \to \infty$.

LEMMA 2. *When $\Lambda \to \infty$, $\rho_\Lambda{}^+(X)$ and $\rho_\Lambda{}^-(X)$ tend to limits $\rho^+(X)$ and $\rho^-(X)$ defining translation invariant equilibrium states ρ^+ and ρ^-. If $\rho^+ = \rho^-$, then*

$$\lim_{\Lambda\to\infty} \rho_\Lambda(X) = \rho^+(X) = \rho^-(X). \tag{A4}$$

The existence of the limits follows from the monotonicity (A3). Uniqueness of the limits implies their translation invariance. Finally, if $\rho^+ = \rho^-$, (A2) implies (A4).

LEMMA 3. *If* (A1) *holds there exists only one translation invariant equilibrium state for the interaction Φ.*

It suffices to prove that the graph of the *pressure P* has a unique tangent plane at $(\Phi, P(\Phi))$ (see [8]), or that the function $\lambda \to P(\Phi + \lambda\Psi)$ is differentiable at $\lambda = 0$ for all translationally invariant interactions Ψ with finite range. We have

$$P(\Phi + \lambda\Psi) = \lim_{\Lambda\to\infty} |\Lambda|^{-1} \log Z_\Lambda{}^\lambda(z),$$

where

$$Z_\Lambda{}^\lambda(z) = \sum_{X\subset\Lambda} z^{|X|} \exp\left[\frac{1}{2}\beta \sum_{x\in X}\sum_{y\in\Lambda\setminus X} \varphi(x-y) - \lambda \sum_{Y\subset X} \Psi(Y)\right],$$

$$z = \exp\left[\beta\left(\mu - \frac{1}{2}\sum_{x\neq 0}\varphi(x)\right)\right].$$

Using the method of [10][4] it is seen that, given $\epsilon > 0$, there exists $\delta > 0$ such that if $|\lambda| < \delta$ (λ complex), then $Z_\Lambda{}^\lambda$ does not vanish in the region $\{z \in \mathbf{C}: ||z| - 1| < \epsilon\}$. Therefore, for real $z \neq 1$, $P(\Phi + \lambda\Psi)$ is analytic in λ at $\lambda = 0$ and the lemma is proved.

Proof of the Theorem. Since (A1) holds by assumption, Lemma 3 shows that

[3] See Griffiths [5], Kelly and Sherman [7], Ginibre [4].

[4] Based on an idea of Asano [1].

there is only one translation invariant equilibrium state ρ. In particular we have $\rho^+ = \rho^- = \rho$, and Lemma 2 gives

$$\lim_{\Lambda\to\infty} \rho_\Lambda(X) = \rho(X)$$

independently of boundary conditions. Therefore, there is only one equilibrium state.

Remark. Also for $\mu = \frac{1}{2}\sum_{x\neq 0} \varphi(x)$, the presence of a unique invariant equilibrium state implies that there is only one equilibrium state. This situation prevails for small β (large temperature).

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